International Journal of Theoretical Physics, Vol. 38, No. 10, 1999

Topology of the Symmetry Group of the Standard Model

M. A. Aguilar[1] **and M. Socolovsky**[2]

Received November 16, 1998

We study the topological structure of the symmetry group of the standard model, $G_{SM} = U(1) \times SU(2) \times SU(3)$. Locally, $G_{SM} \cong S^1 \times (S^3)^2 \times S^5$. For $SU(3)$, which is an S^3-bundle over S^5 (and therefore a local product of these spheres) we give a canonical gauge i.e., a canonical set of local trivializations. These formulas give explicitly the matrices of $SU(3)$ without using the Lie algebra (Gell-Mann matrices). Globally, we prove that the characteristic function of $SU(3)$ is the suspension of the Hopf map $S^3 \xrightarrow{h} S^2$. We also study the case of $SU(n)$ for arbitrary n, in particular the cases of $SU(4)$, a flavor group, and of $SU(5)$, a candidate group for grand unification. We show that the 2-sphere is also related to the fundamental symmetries of nature due to its relation to $SO^0(3, 1)$, the identity component of the Lorentz group, a subgroup of the symmetry group of several gauge theories of gravity.

1. INTRODUCTION

A typical gauge theory on a spacetime M^4 is a theory of connections on a principal G-bundle ξ over M^4, where G is the symmetry group. The connections are coupled to matter fields, which are sections of associated bundles of ξ, defined by linear representations of G.

As is well known, the symmetry group of the electroweak and strong forces (standard model) before spontaneous symmetry breaking is given by (Taylor, 1976)

$$G_{SM} = U(1) \times SU(2) \times SU(3)$$

[1] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Cd. Universitaria, 04510 México, D.F., México; e-mail: marcelo@math.unam.mx.
[2] Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Circuito Exterior, Cd. Universitaria, 04510 México, D.F., México; e-mail: socolovs©nuclecu.unam.m x.

0020-7748/99/1000-248 5$16.00/0

After symmetry breaking, however, G_{SM} breaks down to $G'_{SM} = U(1) \times SU(3)$, the remaining exact symmetry of the electromagnetic and color forces.

In this paper, we study $U(1)$, $SU(2)$, and $SU(3)$ as total spaces of certain principal bundles. In particular, for the case of $SU(3)$, this allows us to give explicit expressions of the matrices of the group without the use of the exponential map applied to linear combinations of the Gell-Mann matrices (Gell-Mann and Ne'eman, 1964). Moreover, we show that the characteristic function of $SU(3)$ is the suspension of the Hopf map $S^3 \xrightarrow{h} S^2$ and, as suggested to us by G. Naber (Naber, 1998), this fact might be related to the existence of smooth magnetic monopoles in gauge theories. We also show that as an $SU(n-1)$-bundle, the group $SU(n)$ has a reduction to $SU(n-2)$ when n is even, and has no such reduction when n is odd ($n \geq 3$). This bundle reduction is similar to that which occurs in the Higgs mechanism in the context of spontaneous symmetry breaking (Choquet-Bruhat *et al.*, 1989).

The group $U(1)$ is the circle or 1-*sphere* S^1, the unit complex numbers, which is the total space of the real Hopf bundle

$$S^0 \to S^1 \xrightarrow{\kappa_2^{\mathbb{R}}} S^1$$

The group $SU(2)$, given by all complex matrices

$$A = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \qquad \text{with} \quad \det A = 1$$

is the 3-*sphere* S^3 or unit quaternions, since if $z = \alpha + i\beta$ and $w = \gamma + i\delta$, then the condition of unit determinant is $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$. $SU(2)$ is the total space of the complex Hopf bundle

$$S^1 \to S^3 \xrightarrow{\kappa_2^{\mathbb{C}}} S^2$$

In the mathematical literature it is well known that for all $n = 2, 3, \ldots$ the groups $SU(n)$ are principal $SU(n-1)$-bundles over the $(2n-1)$-spheres, i.e., that one has the pairs of maps (Steenrod, 1951)

$$SU(n-1) \xrightarrow{\iota} SU(n) \xrightarrow{\pi_n} S^{2n-1}$$

where ι and π are the canonical inclusion and projection, respectively. In particular, for $n = 3$ one has the $SU(2)$-bundle

$$SU(2) \to SU(3) \xrightarrow{\pi_3} S^5$$

which in particular means that locally $SU(3) \cong S^5 \times S^3$ since $SU(2) \cong S^3$; moreover, according to the theory of bundles, the isomorphism classes of

$SU(2)$-bundles over S^5 are in one-to-one correspondence with the fourth homotopy group of $SU(2)$, i.e., $k_{SU(2)}(S^5) \leftrightarrow \Pi_4(SU(2)) \cong \Pi_4(S^3) \cong \mathbb{Z}_2 = \{0, 1\}$: 0 corresponds to the trivial bundle, $S^5 \times S^3$, while 1 corresponds to $SU(3)$ (see Section 3.2). In other words, $SU(3)$, the symmetry group of the strong interactions, *is the unique* (up to isomorphism) *nontrivial SU* (2)-*bundle over the* 5-*sphere*, and as this result shows, it is also constructed from spheres, though not globally. This means that

$$G_{SM} \underset{\text{loc.}}{=} S^1 \times (S^3)^2 \times S^5$$

and, after symmetry breaking

$$G'_{SM} \underset{\text{loc.}}{=} S^1 \times S^3 \times S^5$$

For higher n, however, uniqueness is lost since, for example, for $n = 4$ and $n = 5$ one has the bundles

$$SU(3) \rightarrow SU(4) \overset{\pi_4}{\rightarrow} S^7$$

and

$$SU(4) \rightarrow SU(5) \overset{\pi_5}{\rightarrow} S^9$$

respectively, and $k_{SU(3)}(S^7) \leftrightarrow \Pi_6(SU(3)) \cong \mathbb{Z}_6$ and $k_{SU(4)}(S^9) \leftrightarrow \Pi_8(SU(4)) \cong \mathbb{Z}_{24}$ (Itô, 1993). Notice, however, that locally any $SU(n)$ is a topological product of odd-dimensional spheres:

$$SU(4) \underset{\text{loc.}}{=} S^7 \times SU(3) \underset{\text{loc.}}{=} S^7 \times S^5 \times S^3$$

$$SU(5) \underset{\text{loc}}{=} S^9 \times SU(4) \underset{\text{loc.}}{=} S^9 \times S^7 \times S^5 \times S^3$$

$$\ldots$$

$$SU(n) \underset{\text{loc.}}{=} S^{2n-1} \times S^{2n-3} \times \ldots \times S^5 \times S^3$$

This expression allows us to define a formula which gives, in a canonical way, any element of $SU(n)$ in terms of points of spheres.

A different approach to the study of geometrical aspects of the standard model and general relativity is that of Saller (1998), who considers matter fields as sections of associated bundles defined by (nonlinear) representations on coset spaces of Lie groups. In general relativity, we study the Lie group isomorphism between the proper orthochronous Lorentz group and the group of conformal maps of the 2-sphere, while Saller studies the coset space $GL_4(\mathbb{R})/O(3, 1)$ as a parameter space for the Lorentzian metrics on spacetime.

In Section 2 we briefly review the global construction of the bundle $SU(2) \to SU(3) \xrightarrow{\pi_3} S^5$ and construct a canonical set of local trivializations of $SU(3)$, starting from the (canonical) homogeneous coordinates on $\mathbb{C}P^2$, the complex projective plane. These formulas exhibit $SU(3)$ as a local product of spheres and, moreover, give explicit expressions for all the matrices of $SU(3)$ in terms of the spheres S^5 and S^3.

The above choice of coordinates is natural since for all $n \geq 2$, S^{2n-1} is a principal bundle over $\mathbb{C}P^{n-1}$ with fiber S^1 (complex Hopf bundles):

$$\begin{array}{ccccc} & & & & S^1 \\ & & & & \downarrow \\ SU(n-1) & \to & SU(n) & \xrightarrow{\pi_n} & S^{2n-1} \\ & & & & \downarrow \kappa_n^{\mathbb{C}} \\ & & & & \mathbb{C}P^{n-1} \end{array}$$

and $\mathbb{C}P^{n-1}$ has n canonical charts defining its homogeneous coordinates. Then the bundle π_n, for all n, can be locally trivialized in a canonical way, n being the number of local trivializations. In the following the Hopf map $\kappa_2^{\mathbb{C}}$ will be denoted by h. If $\binom{z}{w} \in S^3 \subset \mathbb{C}^2$ ($|z|^2 + |w|^2 = 1$) and $\mathbb{C}P^1$ is identified with the Riemann sphere $\mathbb{C} = \mathbb{C} \cup \{\infty\}$, then h is given by

$$h\begin{pmatrix} z \\ w \end{pmatrix} = \begin{cases} z/w, & w \neq 0 \\ \infty, & w = 0 \end{cases}$$

It can be proved that h is *essential*, i.e., it is not homotopic to a constant map (Spanier, 1966).

In Section 3 we prove that the characteristic (or clutching) map of $SU(3)$ is the suspension of the Hopf map $S^3 \xrightarrow{h} S^2$. Since the clutching map allows to construct the bundle, then $SU(3)$ is built from information contained in the Hopf map. This map, besides having great importance in homotopy theory, plays a relevant rôle in physics, e.g., in the geometrical description of the spin-1/2 system (Ashtekar and Schilling, 1995; Corichi and Ryan, 1997) and the Dirac monopole of unit magnetic charge (Wu and Yang, 1975).

In Section 4 we investigate the general case of $SU(n)$ and, using a result of Steenrod for $U(n)$, we prove that for even n, $n \geq 2$, the characteristic map g_{n+1}: $S^{2n} \to SU(n)$ of $SU(n+1)$ is a homotopy lifting of the $(2n-3)$th suspension of h. (If $Z \xrightarrow{p} Y$ is a projection and $X \xrightarrow{f} Y$ is a continuous funtion, then p lifts f if there is a continuous function $X \xrightarrow{g} Z$ such that $p \circ g = f$. The lifting is up to homotopy if $p \circ g \sim f$.) On the other hand, for odd n, $n \geq 3$, $\pi_n \circ g_{n+1}$ is inessential or *nullhomotopic* (i.e., homotopic to a constant map). From the physical point of view, the cases of $SU(5)$ and $SU(4)$ are particularly interesting, since $SU(5)$ is a candidate group for grand unification (Mohapatra, 1986), and $SU(4)$ is a flavor group.

Using the concept of bundle reduction (see, e.g., Aguilar and Socolovsky, 1997a), we give a geometric interpretation to these results, namely if n is odd, then $SU(n)$ as an $SU(n-1)$-bundle does not have a reduction to $SU(n-2)$, and if n is even, then it has such reduction. These reductions refer to the internal structure of the symmetry groups, while those associated with the Higgs mechanism (Choquet-Bruhat *et al.*, 1989) are related to the geometry of the principal bundles describing the gauge theories.

In Section 5 we briefly discuss how the 2-*sphere* S^2 appears in the context of the symmetry group of the fundamental interactions due to the canonical isomorphism between its conformal group and $SO^0(3, 1)$, the proper orthochronous Lorentz group.

2. THE BUNDLE $S^3 \to SU(3) \xrightarrow{\pi_3} S^5$

2.1. The Groups $U(3)$ and $SU(3)$

The n-dimensional complex vector space $\mathbb{C}^n$ equipped with the Hermitian scalar product $\langle z, w \rangle = \Sigma_{i=1}^{n} \bar{z}_i w_i$ is a Hilbert space. The $n \times n$ complex matrices which leave $\langle , \rangle$ invariant form the group $U(n)$, i.e., $U(n) = \mathrm{Aut}\,(\mathbb{C}^n, \langle , \rangle)$: *the group of automorphisms of* $\mathbb{C}^n$ *as a Hilbert space*. If $A \in U(n)$ and A^* is the transpose conjugate matrix, then $A^*A = I$, i.e., $A^* = A^{-1}$, so $|\det A| = 1$ and $\dim_{\mathbb{R}} U(n) = n^2$. The topology of $U(n)$ is inherited from the vector space of $n \times n$ complex matrices, which is isomorphic to Euclidean space E^{2n^2}. $U(n)$ is a Lie group and $SU(n)$ is the closed Lie subgroup consisting of matrices whose determinant is 1. Since $U(n)$ is compact, $SU(n)$ is also compact.

For $n = 3$, $SU(3)$ is 2-connected, i.e., $\Pi_k(SU(3)) = 0$ for $k = 1, 2$, and $\Pi_3(SU(3)) \cong \mathbb{Z}$. Topologically, $U(3) \cong SU(3) \times U(1)$, then $U(3)$ is connected but not 1-connected since $\Pi_1(U(3)) \cong \mathbb{Z}$.

2.2. The Inclusion and Action $SU(2) \to SU(3)$

Let

$$\iota:\quad SU(2) \to SU(3),\qquad \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \mapsto \begin{pmatrix} z & w & 0 \\ -\bar{w} & \bar{z} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be the inclusion of $SU(2)$ into $SU(3)$; call $SU(2)' = \iota(SU(2))$. Clearly $SU(2)' \cong SU(2)$, both topologically and as a group. The right action $SU(3) \times SU(2)' \to SU(3)$ is given by matrix multiplication $(B, A) \mapsto BA$, i.e.,

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \sigma & \varepsilon & \varphi \\ \kappa & \lambda & \mu \end{pmatrix}\begin{pmatrix} z & w & 0 \\ -\bar{w} & \bar{z} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha z - \beta\bar{w} & \alpha w + \beta\bar{z} & \gamma \\ \delta z - \varepsilon\bar{w} & \delta w + \varepsilon\bar{z} & \varphi \\ \kappa z - \lambda\bar{w} & \kappa w + \lambda\bar{z} & \mu \end{pmatrix}$$

Let q: $SU(3) \to SU(3)/SU(2)'$ be the quotient map, i.e., $q(B) = [B]$, where $SU(3)/SU(2)'$ is the orbit space $\{[B]\}_{B\in SU(3)}$ with the quotient topology, and $[B] = BSU(2)'$, in particular, $[I] = SU(2)'$. Notice that $|\gamma|^2 + |\varphi|^2 + |\mu|^2 = 1$, i.e.,

$$\begin{pmatrix} \gamma \\ \varphi \\ \mu \end{pmatrix} \in S^5 \subset \mathbb{C}^3$$

It is easy to verify that the following diagram commutes:

$$\begin{array}{ccc} & SU(3) & \\ {}^{g}\swarrow & & \searrow^{\pi_3} \\ SU(3)/SU(2)' & \xrightarrow{\kappa} & S^5 \end{array}$$

where

$$\pi_3(B) = \begin{pmatrix} \gamma \\ \varphi \\ \mu \end{pmatrix} \qquad \text{and} \qquad \kappa([B]) = B\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In particular,

$$\kappa([I]) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

κ turns out to be a *homeomorphism* with inverse

$$\kappa^{-1}\begin{pmatrix}\gamma\\ \varphi\\ \mu\end{pmatrix} = B'SU(2)' \qquad \text{for any} \quad B' = \begin{pmatrix}\cdots & \gamma\\ \cdots & \varphi\\ \cdots & \mu\end{pmatrix} \in SU(3)$$

(an explicit formula for κ will be given in Section 2.3). Clearly

$$\pi_3^{-1}\left(\left\{\begin{pmatrix}\gamma\\ \varphi\\ \mu\end{pmatrix}\right\}\right) = B'SU(2)' \cong SU(2)' \cong SU(2) \cong S^3$$

so S^3 is the fiber of π_3.

2.3. Local Trivializations

Consider the S^1-bundle $S^5 \overset{\kappa_3^{\mathbb{C}}}{\rightarrow} \mathbb{C}P^2$, where the complex projective plane is the space of complex lines through the origin in $\mathbb{C}^3$; $\mathbb{C}P^2$ has three canonical charts given by the open sets

$$V_k = \left\{ \bar{z}(\mathbb{C}\setminus\{0\}) \middle| \bar{z} = \begin{pmatrix} z_1\\ z_2\\ z_3\end{pmatrix} \text{ and } z_k \neq 0 \right\} \subset \mathbb{C}P^2, \text{ for } k = 1, 2, 3$$

and the homeomorphisms $V_k \rightarrow \mathbb{C}^2$ map $\bar{z}(\mathbb{C}\setminus\{0\})$ to $(\xi_i, \xi_j) = (z_i/z_k, z_j/z_k)$ with i, j, k in cyclic order, the ξ_i are called homogeneous coordinates. Then the preimages of V_k by the projection κ_3 define three open sets in S^5 given by

$$U_k = \kappa_3^{-1}(V_k) = \left\{ \begin{pmatrix} z_1\\ z_2\\ z_3\end{pmatrix} \middle| z_k \neq 0 \right\} \equiv S_k^5 \subset S^5$$

$\cup_{i=1}^3 U_i = S^5$ with $U_i \cap U_j \neq \phi$ for all i, j and $(0, 0, 1) \in U_3 \subset S^5$ but $(0, 0, 1) \notin U_1, U_2$. Notice that the complements of S_k^5 with respect to S^5 are homeomorphic to S^3:

$$(S_3^5)^c = S^5 \setminus S_3^5 = \left\{ \begin{pmatrix} z_1\\ z_2\\ 0\end{pmatrix} \middle| |z_1|^2 + |z_2|^2 = 1 \right\} \cong S^3$$

and analogous formulas for S_1^5 and S_2^5; $(S_k^5)^c$ are closed sets in S^5. We shall trivialize the $SU(3) \overset{\pi_3}{\rightarrow} S^5$ bundle over the U_k.

In order to construct local sections of the bundle π_3, consider the following complex matrices:

$$\tilde{C} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 1 & 0 & a \\ 0 & 0 & b \\ 0 & 1 & c \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}$$

It is easy to verify that the three column vectors in each of them are linearly independent if, respectively, $c \neq 0$, $b \neq 0$, and $a \neq 0$. In the three cases we take $|a|^2 + |b|^2 + |c|^2 = 1$. By the Gram–Schmidt procedure we can construct unitary matrices $\hat{C}$, $\hat{B}$, and $\hat{A}$ and then, multiplying each of them from the right by the matrix $\begin{pmatrix} z & 0 \\ 0 & I \end{pmatrix}$, where $z^{-1} = det\hat{C}$, $det\hat{B}$, or $det\hat{A}$, we obtain the following elements of $SU(3)$:

$$C = \begin{pmatrix} \frac{|c|^2}{c\sqrt{1-|b|^2}} & \frac{-a\bar{b}}{\sqrt{1-|b|^2}} & a \\ 0 & \sqrt{1-|b|^2} & b \\ \frac{-\bar{a}}{\sqrt{1-|b|^2}} & \frac{-\bar{b}c}{\sqrt{1-|b|^2}} & c \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{-|b|^2}{b\sqrt{|a|^2+|b|^2}} & \frac{a\bar{c}}{\sqrt{1-|c|^2}} & a \\ \frac{-\bar{a}}{\sqrt{|a|^2+|b|^2}} & \frac{-b\bar{c}}{\sqrt{1-|c|^2}} & b \\ 0 & \sqrt{1-|c|^2} & c \end{pmatrix}$$

$$A = \begin{pmatrix} \frac{-\bar{b}}{\sqrt{1-|c|^2}} & \frac{-a\bar{c}}{\sqrt{1-|c|^2}} & a \\ \frac{|a|^2}{a\sqrt{1-|c|^2}} & \frac{-b\bar{c}}{\sqrt{1-|c|^2}} & b \\ 0 & \sqrt{1-|c|^2} & c \end{pmatrix}$$

with $det\hat{C} = c/|c|$, $det\hat{B} = -b/|b|$, and $det\hat{A} = a/|a|$.

(These formulas give an explicit expression for B' in κ^{-1} of Section 2.2: given

$$\begin{pmatrix} \gamma \\ \varphi \\ \mu \end{pmatrix} \in S^5$$

we choose B' equal to A, B, or C if, respectively, γ, φ, or μ is $\neq 0$.) We define local sections σ_k: $S^5_k \rightarrow SU(3)$ as follows:

$$\sigma_1: \quad S^5_1 \rightarrow SU(3), \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \sigma_1 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A$$

$$\sigma_2: \quad S^5_2 \rightarrow SU(3), \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \sigma_2 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = B$$

and

$$\sigma_3: \quad S^5_3 \rightarrow SU(3), \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \sigma_3 \begin{pmatrix} a \\ b \\ c \end{pmatrix} = C$$

If π: $P \rightarrow X$ is a principal. G-bundle over X, and $\sigma_\beta : U_\beta \rightarrow P$ are local sections, then φ_β: $\pi^{-1}(U_\beta) \equiv P_\beta \rightarrow U_\beta \times G$, where $\varphi_\beta(a) = (x, \gamma_\beta(a))$ with $x = \pi(a)$ and $a = \sigma_\beta(\pi(a)) \cdot \gamma_\beta(a)$, are local trivializations. If φ_α and φ_β are local trivializations and $U_\alpha \cap U_\beta \neq \phi$, then $\varphi_\beta \circ \varphi_\alpha^{-1}$: $(U_\alpha \cap U_\beta) \times G \rightarrow (U_\alpha \cap U_\beta) \times G$ satisfies $\varphi_\beta \circ \varphi_\alpha^{-1}(x, g) = (x, g_{\beta\alpha}(x))$, where $g_{\beta\alpha}$: $U_\alpha \cap U_\beta \rightarrow G$ are the *transition functions* and $\sigma_\beta(x) \cdot g_{\beta\alpha}(x) = \sigma_\alpha(x)$. In our case, with $G = SU(2)$, $P = SU(3)$, $X = S^5$, $\beta = k = 1, 2, 3$, $U_k = S^5_k$, and

$$P_k = SU(3)_k = \left\{ \begin{pmatrix} \cdots & z_1 \\ \cdots & z_2 \\ \cdots & z_3 \end{pmatrix} \in SU(3) \middle| z_k \neq 0 \right\}$$

the local trivilizations of the $SU(2)$-bundle $p \equiv \pi_3$: $SU(3) \rightarrow S^5$ are

$$\varphi_1: \quad SU(3)_1 \rightarrow S^5_1 \times SU(2)',$$

$$\varphi_1(R) = (p(R), (\sigma_1(p(R)))^{-1}R) = \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, A^*R \right)$$

$$\varphi_2: \quad SU(3)_2 \rightarrow S^5_2 \times SU(2)',$$

$$\varphi_2(S) = (p(S), (\sigma_2(p(S)))^{-1}S) = \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, B^*S \right)$$

and

$$\varphi_3: \quad SU(3)_3 \rightarrow S_3^5 \times SU(2)',$$

$$\varphi_3(T) = (p(T), (\sigma_3(p(T)))^{-1}\, T) = \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, C^*T \right)$$

The matrices A^*R, B^*S, and C^*T are of the form

$$\begin{pmatrix} \multicolumn{2}{c}{\mathrm{D}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{with} \quad D \in SU(2)$$

φ_1, φ_2, and φ_3 exhibit the local structure of $SU\,(3)$. The transition functions are

$$g_{12}: \quad S_1^5 \cap S_2^5 \rightarrow SU(2)', \qquad g_{12} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = A^*B$$

$$g_{23}: \quad S_2^5 \cap S_3^5 \rightarrow SU(2)', \qquad g_{23} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = B^*C$$

and

$$g_{31}: \quad S_3^5 \cap S_1^5 \rightarrow SU(2)', \qquad g_{31} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = C^*A$$

Notice that $I \in SU(3)_3$, but $I \notin SU(3)_j$ for $j = 1, 2$, so $SU(3)_3$ is an open neighborhood of the identity. The inverses of the local trivializations are given by

$$\psi_3: \quad S_3^5 \times SU(2) \rightarrow SU(3)_3, \qquad \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, R_3 \right) \mapsto CR_3'$$

$$\psi_2: \quad S_2^5 \times SU(2) \rightarrow SU(3)_2, \qquad \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, R_2 \right) \mapsto BR_2'$$

and

$$\psi_1: \quad S_1^5 \times SU(2) \to SU(3)_1, \qquad \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}, R_1 \right) \mapsto AR_1'$$

with $\psi_i = \varphi_i^{-1}$ after identifying $SU(2) \cong SU(2)'$, and

$$R_k' = \begin{pmatrix} R_k & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad k = 1, 2, 3$$

These formulas give *all* elements of $SU(3)$ in terms of points of the 3- and 5-spheres.

With the help of the above formulas the set of matrices of $SU(3)$ can be divided into seven disjoint subsets: $SU(3)_{123}$, $SU(3)_{i,jk}$, and $SU(3)_{ij,k}$, respectively, the pieces of $SU(3)$ lying over $S_{123}^5 = S_1^5 \cap S_2^5 \cap S_3^5$, $S_{i,jk}^5 = S^5 \backslash (S_j^5 \cup S_k^5)$, and $S_{ij,k}^5 = S_i^5 \cap S_j^5 \backslash S_{123}^5$, with $i, j, k \in \{1, 2, 3\}$ in cyclic order:

$$\begin{pmatrix} \dfrac{|c|^2 z/c + a\bar{b}\bar{w}}{\sqrt{1 - |b|^2}} & \dfrac{|c|^2 w/c - a\bar{b}\bar{z}}{\sqrt{1 - |b|^2}} & a \\ -\bar{w}\sqrt{1 - |b|^2} & \bar{z}\sqrt{1 - |b|^2} & b \\ \dfrac{-\bar{a}z + \bar{b}c\bar{w}}{\sqrt{1 - |b|^2}} & -\dfrac{\bar{a}w + \bar{b}c\bar{z}}{\sqrt{1 - |b|^2}} & c \end{pmatrix} \in SU(3)_{123}$$

$|a|^2 + |b|^2 + |c|^2 = 1; \ 0 < |a|, |b|, |c| < 1;$

$$\begin{pmatrix} 0 & 0 & e^{i\varphi} \\ ze^{-i\varphi} & we^{-i\varphi} & 0 \\ -\bar{w} & \bar{z} & 0 \end{pmatrix} \in SU(3)_{1,23}$$

$$\begin{pmatrix} -ze^{-i\varphi} & -we^{-i\varphi} & 0 \\ 0 & 0 & e^{i\varphi} \\ -\bar{w} & \bar{z} & 0 \end{pmatrix} \in SU(3)_{2,31}$$

$$\begin{pmatrix} ze^{-i\varphi} & we^{-i\varphi} & 0 \\ -\bar{w} & \bar{z} & 0 \\ 0 & 0 & e^{i\varphi} \end{pmatrix} \in SU(3)_{3,12}$$

$$\begin{pmatrix} -z\bar{b} & -w\bar{b} & a \\ \frac{z|a|^2}{a} & \frac{w|a|^2}{a} & b \\ -\bar{w} & \bar{z} & 0 \end{pmatrix} \in SU(3)_{12,3}$$

$|a|^2 + |b|^2 = 1,\ a,\ b \neq 0;$

$$\begin{pmatrix} \frac{-z|b|}{b} & \frac{-w|b|}{b} & 0 \\ \frac{\bar{w}b\bar{c}}{|b|} & \frac{-\bar{z}b\bar{c}}{|b|} & b \\ -|b|\bar{w} & |b|\bar{z} & c \end{pmatrix} \in SU(3)_{23,1}$$

$|b|^2 + |c|^2 = 1,\ b,\ c \neq 0;$

$$\begin{pmatrix} \frac{|c|^2 z}{c} & \frac{|c|^2 w}{c} & a \\ -\bar{w} & \bar{z} & 0 \\ -\bar{a}z & -\bar{a}w & c \end{pmatrix} \in SU(3)_{31,2}$$

$|a|^2 + |c|^2 = 1;\ a,\ c \neq 0;$ with $|z|^2 + |w|^2 = 1$ and $\varphi \in [0, 2\pi)$.

Remark. The above results can be extended to the bundles

$$U(n-1) \to U(n) \xrightarrow{p_n} S^{2n-1}$$

i.e., to the *unitary groups*. In particular, for $n = 3$ we have the pair of maps

$$U(2) \xrightarrow{\iota} U(3) \xrightarrow{p_3} S^5$$

with

$$U(2) = \left\{ \begin{pmatrix} z & w \\ \bar{w}e^{i\lambda} & -\bar{z}e^{i\lambda} \end{pmatrix} \middle| |z|^2 + |w|^2 = 1,\ \lambda \in [0, 2\pi) \right\}$$

The local trivializations of $U(3)$, ϕ_k: $U(3)_k \to S^5_k \times U(2)'$, $k = 1, 2, 3$, and where $U(2)' = \iota(U(2))$ are given by the same formulas as those for $SU(3)$, with the matrices A, B, and C, respectively, replaced by the matrices $\hat{A}$, $\hat{B}$, and $\hat{C}$.

3. $SU(3)$ FROM THE $N = 2$ HOPF BUNDLE

3.1. Suspension

The *suspension* of a topological space X is the quotient space given by

$$SX = \frac{X \times I}{X \times \{0\},\ X \times \{1\}} = \{[x, t]\}_{(x,t)\in X\times I}$$

with

$$[x, t] = \begin{cases} \{(x, t)\}, & t \in (0, 1) \\ X \times \{0\}, & t = 0 \\ X \times \{1\}, & t = 1 \end{cases}$$

This means that in the product $X \times I$, $X \times \{0\}$ has been identified to one point and $X \times \{1\}$ has been identified to another point. Intuitively, it is clear that $SS^0 \cong S^1$, $SS^1 \cong S^2, \ldots, SS^{n-1} \cong S^n$. The suspension of a continuous function is defined by $Sf([x, t]) = [f(x), t]$, which satisfies the *functorial* properties $Sid_X = id_{SX}$ and $S(g \circ f) = Sg \circ Sf$ if f: $X \to Y$ and g: $Y \to Z$. If p_Z: $Z \times I \to SZ$ is the projection $p(z, t) = [z, t]$, then the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \iota_0\downarrow & & \downarrow\iota_0 \\ X \times I & \xrightarrow{f\times id} & Y \times I \\ p_X\downarrow & & \downarrow p_Y \\ SX & \xrightarrow{Sf} & SY \end{array}$$

If H: $X \times I \to Y$ is a homotopy between h_0 and h_1, then SH:$SX \times I \to SY$ given by $SH([x, t], t') = [H(x, t'), t]$, i.e., $(SH)_{t'} = SH_{t'}$ is a homotopy between Sh_0 and Sh_1. SH is called the *suspension of the homotopy*. Then there is a well-defined function between homotopy classes of maps S: $[X, Y] \to [SX, SY]$, $[f] \to S([f]) := [Sf]$.

If X is a pointed space with base point x_0, then the *reduced suspension* of X, $S_r X$, is defined by

$$S_r X = \frac{X \times I}{X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I}$$

i.e., all the points in $X \times \{0\}$, $X \times \{1\}$, and $\{x_0\}$ are identified to one point. In this case its elements are given by

$$[x,t]=\begin{cases} X\times\{0\}\cup X\times\{1\}\cup\{x_0\}\times I, & x=x_0,\quad \text{all}\quad t\in I \\ & t=0 \text{ or } 1,\quad \text{all}\quad x\in X \\ \{(x,t)\}, & t\in(0,1)\quad\text{and}\quad x\neq x_0 \end{cases}$$

$\tilde{x}_0=[x_0,t]$ is the base point of S_rX. If $f\colon X\to Y$ preserves the base points, i.e., if $f(x_0)=y_0$, then $S_r f(\tilde{x}_0)=y_0$, and if $h_0\overset{H}{\sim}h_1(rel\ x_0)$, then $Sh_0\overset{SH}{\sim}Sh_1(rel\ \tilde{x}_0)$. ($rel\ x_0$ means that the homotopy H preserves the base point.) Also, there is a homeomorphism $\varphi_{n+1}^{-1}\colon S_rS^n\to S^{n+1}$ given by

$$\varphi_{n+1}^{-1}([\bar{x},t])=\begin{cases} p_-^{-1}(2t\bar{x}+(1-2t)\bar{x}_0), & t\in[0,1/2] \\ p_+^{-1}((2-2t)\bar{x}+(2t-1)\bar{x}_0), & t\in[1/2,1] \end{cases}$$

where

$$\bar{x}_0=(1,0,\ldots,0)\in S^{n+1}=\left\{(x_1,\ldots,x_{n+2})\Big|\ \sum_{i=1}^{n+2}x_i^2=1\right\}\subset\mathbb{R}^{n+2}$$

is the base point, $S^n=\{\bar{x}\in S^{n+1}|x_{n+2}=0\}$, and if $H_+=\{\bar{x}\in S^{n+1}|x_{n+2}\geq 0\}$, $H_-=\{\bar{x}\in S^{n+1}|x_{n+2}\leq 0\}$, and $D^{n+1}=\{(x_1,\ldots,x_{n+1},0)|\ \Sigma_{i=1}^{n+1}x_i^2\leq 1\}$, then $p_{+(-)}\colon H_{+(-)}\to D^{n+1}$ are the homeomorphisms given by $(x_1,\ldots,x_{n+2})\mapsto(x_1,\ldots,x_{n+1},0)$ with inverses

$$(x_1,\ldots,x_{n+1},0)\mapsto\left(x_1,\ldots,x_{n+1},+(-)\sqrt{1-\sum_{i=1}^{n+1}x_i^2}\right)$$

respectively (Spanier, 1966). In particular,

$$\tilde{\bar{x}}_0=S^n\times\{0\}=S^n\times\{1\}=\{\bar{x}_0\}\times I\overset{\varphi_{n+1}^{-1}}{\to}\bar{x}_0$$

and if $\bar{x}\neq\bar{x}_0$, then

$$[\bar{x},1/2]=\{(\bar{x},1/2)\}\overset{\varphi_{n+1}^{-1}}{\to}\bar{x}$$

The inverse homeomorphism is given by the following formulas: $\bar{x}_0\mapsto\tilde{\bar{x}}_0$, if $\bar{x}\in S_n$ and $\bar{x}\neq\bar{x}_0$ then $\bar{x}\mapsto[\bar{x},1/2]=\{(\bar{x},1/2)\}$, $(0,\ldots,0,1)=N$ (*north pole*) $\mapsto[-\bar{x}_0,3/4]=\{(-\bar{x}_0,3/4)\}$, $(0,\ldots,-1)=S$ (*south pole*) $\mapsto[-\bar{x}_0,1/4]=\{(-\bar{x}_0,1/4)\}$, and if $\bar{x}\in H_{+(-)}\backslash S^n$, $\bar{x}\neq S,N$, then

$$\varphi_{n+1}(\bar{x})=[\bar{z}(\bar{x}),t_{+(-)}(\bar{x})]=\{(\bar{z}(\bar{x}),t_{+(-)}(\bar{x}))\}$$

with

$$\bar{z}(\bar{x}) = \frac{(2(1 - x_1)x_1 - x_{n+2}^2, 2(1 - x_1)x_2, \ldots, 2(1 - x_1)x_{n+1})}{2(1 - x_1) - x_{n+2}^2}$$

and $t_{+(-)}(\bar{x}) = 1/2 + (-)x_{n+2}^2/[4(1 - x_1)]$.

3.2. *SU*(3) from the Hopf Map *h*

Let G be a path-connected topological group. Then the set of isomorphism classes of principal G-bundles over the n-sphere $k_G(S^n)$ is in one-to-one correspondence with $\Pi_{n-1}(G)$ (Steenrod, 1951). This can be understood from the fact that the n-sphere can be covered by two open sets U_1, U_2, which are homeomorphic to n-balls and contain S^{n-1}, and the fact that any bundle over an n-ball is trivial. Using these trivializations, there is only one transition function g_{12}: $U_1 \cap U_2 \to G$ for a bundle ξ. Then we associate to ξ the map $g_{12}|_{S^{n-1}}$: $S^{n-1} \to G$, called the characteristic map of ξ. Therefore, if the characteristic maps of two bundles are in the same homotopy class, then the corresponding bundles are isomorphic, and a bundle is trivial if and only if its characteristic map is null-homotopic. Notice that in our construction of the local charts for $SU(3)$ we have used a different trivialization. In the following we shall consider the bundles $SU(n - 1) \to SU(n) \xrightarrow{\pi_n} S^{2n-1}$ and call g_n: $S^{2n-2} \to SU(n - 1)$ the corresponding characteristic maps.

To study the case $n = 3$ we need the following:

Proposition. The successive suspensions of the Hopf map, S_rh: $S^4 \to S^3$, S_r^2h: $S^5 \to S^4$, ..., are essential (Steenrod and Epstein, 1962).

As a consequence, we have the following:

Proposition. $SU(3)$ is determined by the suspension of the Hopf map.

Proof. For $n = 3$, $k_{SU(2)}(S^5) \cong [S^4, S^3] \cong \Pi_4(S^3) \cong \mathbb{Z}_2 = \{0, 1\}$ and g_3: $S^4 \to S^3$. By the proposition above, S_rh is essential. To see that g_3 is also essential we will show that the bundle $SU(3) \xrightarrow{\pi_3} S^5$ is not trivial. By (Itô, 1993), $\Pi_4(SU(3)) \cong 0$; on the other hand, $\Pi_4(S^5 \times SU(2)) \cong \Pi_4(S^5) \times \Pi_4(S^3) \cong 0 \times \mathbb{Z}_2 \cong \mathbb{Z}_2$. Hence $SU(3)$ is not isomorphic to the trivial bundle. Since $\Pi_4(S^3) \cong \mathbb{Z}_2$ we have that $[g_3] = [S_rh]$. QED

4. THE CASE OF *SU*(*N*)

4.1. *SU*(4)

For $n = 4$, $k_{SU(3)}(S^7) \cong [S^6, SU(3)] \cong \Pi_6(SU(3)) \cong \mathbb{Z}_6$, which has two generators. This means that up to isomorphism there are five nontrivial $SU(3)$-bundles over S^7, one of them being $SU(4)$ since $\Pi_6(SU(4)) \cong 0$ and $\Pi_6(S^7 \times$

$SU(3)) \cong \Pi_6(S^7) \times \Pi_6(SU(3)) \cong \mathbb{Z}_6$. Let g_4: $S^6 \to SU(3)$ be its characteristic map. If g_4 were a homotopy lifting of S_r^3h, then there should exist a lifting g_4' by π_3: $SU(3) \to S^5$ of S_r^3h: $S^6 \to S^5$, i.e., a commutative diagram

$$\begin{array}{ccc} & SU(3) & \\ {}^{g_4'}\nearrow & & \searrow^{\pi_3} \\ S^6 & \xrightarrow{S_r^3h} & S^5 \end{array}$$

with $g_4' \sim g_4$. We have the following result.

Proposition. π_3 does *not* lift S_r^3h.

Proof. We will show that the homomorphism π_{3*}: $\Pi_6(SU(3)) \to \Pi_6(S^5)$ is zero. This implies that any map $S^6 \xrightarrow{f} S^5$ which factorizes through π_3, i.e., a map for which there exists a map $S^6 \xrightarrow{g} SU(3)$ such that $\pi_3 \circ g \sim f$, is null-homotopic. The result now follows from this since S_r^3h is essential.

Consider the long exact homotopy sequence (Steenrod, 1951) of the principal bundle $SU(2) \to SU(3) \xrightarrow{\pi_3} S^5$:

$$\ldots \to \Pi_6(SU(3)) \xrightarrow{\pi_{3*}} \Pi_6(S^5) \xrightarrow{\delta} \Pi_5(S^3) \xrightarrow{\iota_*} \Pi_5(SU(3)) \to \ldots$$

This gives an exact sequence:

$$\ldots \to \mathbb{Z}_6 \xrightarrow{\beta} \mathbb{Z}_2 \xrightarrow{\gamma} \mathbb{Z}_2 \xrightarrow{\alpha} \mathbb{Z} \to \ldots$$

where we called β, γ, and α the homomorphisms corresponding to π_{3*}, δ, and ι_*, respectively. Since $\mathbb{Z}_2$ is a torsion group and $\mathbb{Z}$ is torsion-free, then the homomorphism α is zero. Therefore γ is an isomorphism, i.e., $ker(\gamma) = ker(\delta) = \{0\} = Im(\pi_{3*})$, i.e., $\pi_{3*} = 0$. QED

Remark. This result can also be obtained from the general theorem proved in Section 4.4. However, the proof given above is simpler.

4.2. (*H*, *f*)-Structures

Let H and G be topological groups (e.g., Lie groups), ξ_H: $H \to PH \xrightarrow{\pi_H} BH$ and ξ_G: $G \to PG \xrightarrow{\pi_G} BG$ their universal bundles, and f: $H \to G$ a topological group homomorphism. Then the action $\bar{f}$: $H \times G \to G, \bar{f}(h, g) = f(h)g$ induces the associated principal G-bundle $(\xi_H)_G$: $G \to PH \times_H G \to BH$ with total space $PH \times_H G = \{[a, g]\}_{(a,g)\in PH\times G}, [a, g] = \{(ah, f(h^{-1})g)\}_{h\in H}$, action $(PH \times_H G) \times G \to PH \times_H G$ given by $[a, g] \cdot g' = [a, gg']$, and projection $(\pi_H)_G([a, g]) = \pi_H(a)$. $PH \times_H G$ is isomorphic to the pullback bundle $(Bf)^*(PG)$, where the induced function Bf: $BH \to BG$ is uniquely defined up to homotopy.

If *HTop* is the category of paracompact topological spaces and homotopy classes of maps and *Set* is the category of sets and functions, then for each topological group K there are two cofunctors k_K and $[\,, BK]$ from *HTop* to *Set* such that for each topological group homomorphism $f\colon H \to G$ there are natural transformations $f_*\colon k_H \to k_G$ and $Bf_*\colon [\,, BH] \to [\,, BG]$, and natural equivalences ψ_H and ψ_G which make the following functorial diagram commutative:

$$\begin{array}{ccc} k_H & \stackrel{f_*}{\rightarrow} & k_G \\ \psi_H\uparrow & & \uparrow\psi_G \\ [\,, BH] & \stackrel{Bf_*}{\rightarrow} & [\,, BG] \end{array}$$

So, for each paracompact topological space X the following set-theoretic diagram commutes:

$$\begin{array}{ccc} k_H(X) & \stackrel{f_*}{\rightarrow} & k_G(X) \\ \psi_H\uparrow\cong & & \cong\uparrow\psi_G \\ [X, BH] & \stackrel{Bf_*}{\rightarrow} & [X, BG] \end{array}$$

where $k_K(X) = \{$isomorphism classes of principal K-bundles over $X\}$, $[X, BK] = \{$unbased homotopy classes of maps from X to $BK\}$, $\psi_K([\alpha]) = [\alpha^*(PK)]$, $f_*([\eta]) = [\xi]$ with $\eta\colon H \to E \stackrel{q}{\to} X$ and $\xi\colon G \to E \times_H G \stackrel{q}{\to} X$, and $Bf_*([\alpha]) = [Bf \circ \alpha]$. If $[\xi] \in k_G(X)$, then $f_*^{-1}(\{[\xi]\})$ is the set of (H, f)-*structures* on ξ; this set can be empty. So, ξ has an (H, f)-structure if and only if there exists a map $\alpha\colon X \to BH$ such that $Bf \circ \alpha \sim F$, where F is the classifying map of ξ. One can show that this definition is equivalent to the existence of a G-bundle isomorphism

$$\begin{array}{ccc} E \times_H G & \stackrel{\bar{\varphi}}{\rightarrow} & P \\ \bar{q}\searrow & & \swarrow\pi \\ & X & \end{array}$$

where $H \to E \stackrel{q}{\to} X$ is a principal H-bundle or, equivalently, to the bundle map

$$\begin{array}{ccc} E \times H & \stackrel{\varphi\times f}{\rightarrow} & P \times G \\ \kappa\downarrow & & \downarrow\psi \\ E & \stackrel{\varphi}{\rightarrow} & P \\ q\searrow & & \swarrow\pi \\ & X & \end{array}$$

where $\varphi = \bar{\varphi} \circ \varphi_f$ with $\varphi_f\colon E \to E \times_H G$ given by $\varphi_f(a) = [a, e]$ (e is the unit of G) (Aguilar and Socolovsky, 1997a). One says that (E, φ) *is an* (H, f)*-structure on* $G \to P \xrightarrow{\pi} X$.

In the case of smooth bundles, if the Lie group homomorphism $H \xrightarrow{f} G$ is an *embedding*, i.e., an injective immersion, then E is called a *reduction* of P to H. In this setting, one has the following:

Proposition. If f is an embedding, then φ is also an embedding.

Proof. Since $\varphi = \bar{\varphi} \circ \varphi_f$ and $\bar{\varphi}$ is a diffeomorphism, then φ is an embedding if and only if φ_f is an embedding; we shall show that φ_f is an embedding. (i) φ_f is injective: Let $\varphi_f(a_1) = \varphi_f(a_2)$, i.e., $[a_1, e] = [a_2, e]$; since $[a, e] = \{(ah, f(h^{-1}))\}_{h \in H}$, then there must exist $h \in H$ such that $(a_1, e) = (a_2 h, f(h^{-1})$, i.e., $a_1 = a_2 h$ and $f(h^{-1}) = e$, but f is injective, so $h^{-1} = h = e'$, the identity in H, and then $a_1 = a_2$.

(ii) $d\varphi_f$ is injective at each $a_0 \in E$: Consider the commutative diagram

$$\begin{array}{ccc} & E \times G & \\ {}^{i}\nearrow & & \searrow^{p} \\ E & \xrightarrow{\varphi_f} & E \times_H G \end{array}$$

where $i(a) = (a, e)$ and $p(a, e) = [a, e]$. By Greub *et al.* (1973), $H \to E \times G \xrightarrow{p} E \times_H G$ is a principal H-bundle, so fixing $(a_0, e) \in E \times G$, there is a map $\alpha_{(a_0,e)}$: $H \to E \times G$, given by $\alpha_{(a_0,e)}(h) = (a_0, e) \cdot h = (a_0 h, f(h^{-1}))$, in particular, $\alpha_{(a_0,e)}(e') = (a_0, e)$. One then has the following diagram of vector spaces:

$$\begin{array}{ccccccc} 0 \to T_{e'}H & \xrightarrow{(d\alpha_{(a_0,e)})_{e'}} & T_{(a_0,e)}(E \times G) & \xrightarrow{(dp)_{(a_0,e)}} & T_{[a_0,e]}(E \times_H G) \to 0 \\ & & {\scriptstyle (di)_{a_0}}\uparrow & \nearrow{\scriptstyle (d\varphi_f)_{a_0}} & \\ & & T_{a_0}E & & \end{array}$$

where the horizontal sequence is exact and the triangle commutes. If p_1 and p_2 are, respectively, the projections of $E \times G$ onto E and G, then $p_1 \circ i(a) = a$, i.e., $p_1 \circ i = id_E$ and $p_2 \circ i(a) = p_2(a, e) = e$, i.e., $p_2 \circ i = \text{const}$, hence $(d(p_1 \circ i))_{a_0} = (d(id_E))_{a_0} = id_{T_{a_0}E}$ and $(d(p_2 \circ i))_{a_0} = (d(\text{const}))_{a_0} = 0$. Therefore, $(di)_{a_0}(v) = ((d(p_1 \circ i))_{a_0}(v), (d(p_2 \circ i))_{a_0}(v)) = (v, 0)$. On the other hand, $p_1 \circ \alpha_{(a_0,e)}(h) = p_1(a_0 h, f(h^{-1})) = a_0 h := \alpha_{a_0}(h)$, $p_2 \circ \alpha_{(a_0,e)}(h) = p_2(a_0 h, f(h^{-1})) = f \circ \gamma(h)$, where γ: $H \to H$ is given by $\gamma(h) = h^{-1}$. Therefore $(d\alpha_{(a_0,e)})_{e'}(w) = ((d(p_1 \circ \alpha_{(a_0,e)}))_{e'}(w)), (d(p_2 \circ \alpha_{(a_0,e)}))_{e'}(w)) = ((d\alpha_{a_0})_{e'}(w), (df)_{e'} \circ ((d\gamma)_{e'}(w)$. Let $(r, s) \in \text{Im}((di)_{a_0}) \cap \text{Im}((d\alpha_{(a_0,e)})_{e'}))$; then $s = 0$ and hence $0 = (df)_{e'}((d\gamma)_{e'}(w))$; therefore $(d\gamma)_{e'}(w) = 0$ because f is an immersion and, since γ is a diffeomorphism, $w = 0$, so $r = (d\alpha_{a_0})_{e'}(0) = 0$, i.e., $\text{Im}((di)_{a_0}) \cap \text{Im}((d\alpha_{(a_0,e)})_{e'}) = \{0\}$. Finally, let $v \in ker((d\varphi_f)_{a_0})$; then $0 =$

$(dp)_{(a_0,e)}((di)_{a_0})(v))$, i.e., $(di)_{a_0}(v) \in ker((dp)_{(a_0,e)}) = \mathrm{Im}((d\alpha_{(a_0,e)})_{e'})$, i.e., $(di)_{a_0}(v) = 0$. Since i is an embedding, then $v = 0$, i.e., $(d\varphi_f)_{a_0}$ is one-to-one. QED

Remark. One often finds in the literature (Kobayashi and Nomizu, 1963; Trautman, 1984) that to define a reduction to a Lie subgroup $H \subset G$, φ is required to be an embedding. The proposition above shows that this is a consequence of the fact that $H \to G$ is an embedding.

4.3. $SU(n) \to SU(n+1) \overset{\pi_{n+1}}{\to} S^{2n+1}$ as an $(SU(n), \iota)$-Structure on $U(n) \to U(n+1) \overset{p_{n+1}}{\to} S^{2n+1}$

Proposition. For $n = 1, 2, 3, \ldots$, the bundle π_{n+1} is a reduction of the bundle p_{n+1}, i.e., one has the $U(n)$-bundle isomorphism given by the commutative diagram

$$\begin{array}{ccc} (SU(n+1) \times_{SU(n)} U(n)) \times U(n) & \overset{\bar{\varphi}\times id}{\to} & U(n+1) \times U(n) \\ \lambda\downarrow & & \downarrow\psi \\ SU(n+1) \times_{SU(n)} U(n) & \overset{\bar{\varphi}}{\to} & U(n+1) \\ q_{n+1}\searrow & & \swarrow p_{n+1} \\ & S^{2n+1} & \end{array}$$

where $\lambda([D, A], B) = [D, AB]$, $q_{n+1}[D, A] = \pi_{n+1}(D) = De_0$, $\psi(C, B) = Cj(B)$, $p_{n+1}(C) = Ce_0$, and $\bar{\varphi}$ and j are given below.

Proof. Consider the inclusion $SU(n+1) \overset{\varphi}{\to} U(n+1)$; one can easily show that this is a smooth bundle map between the principal $SU(n)$-bundle π_{n+1} and the principal $U(n)$-bundle p_{n+1}. Therefore, by Greub *et al.* (1973) the map $\bar{\varphi}$ given by $\bar{\varphi}([D, A]) = Dj(A)$, where j is the inclusion $U(n) \to U(n+1)$ with $j(A) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, is a smooth bundle isomorphism. The inverse of $\bar{\varphi}$ is given as follows: if $C \in U(n+1)$, then $C = Dl(detC)$, where $D = C(l(detC))^{-1} \in SU(n+1)$ and l: $U(1) \to U(n+1)$ is the inclusion $l(z) = \begin{pmatrix} z & 0 \\ 0 & I \end{pmatrix}$, then $[D, l(detC)] = \bar{\varphi}^{-1}(C)$. QED

Remark. Notice that if ι: $SU(n) \to U(n)$ is the inclusion, then $p_n \circ \iota = \pi_n$.

4.4. Proof of the Main Result

Proposition. Let H and G be path-connected topological groups such that each one has the homotopy type of a CW-complex, and let f: $H \to G$ be a continuous homomorphism. Then, in the following diagram each horizontal function is a bijection and each square commutes, for $n = 1, 2, 3, \ldots$:

$$\begin{array}{ccccccc}
[S^{2n}, G] & \overset{\mu_{G\#}}{\rightarrow} & [S^{2n}, \Omega BG] & \overset{adj_{G*}}{\rightarrow} & [S_r S^{2n}, BG] & \overset{\psi_G}{\rightarrow} & k_G(S^{2n+1}) \\
\uparrow f_\# & & \uparrow \Omega Bf_\# & & \uparrow Bf_* & & \uparrow f_* \\
[S^{2n}, H] & \overset{\mu_{H\#}}{\rightarrow} & [S^{2n}, \Omega BH] & \overset{adj_{H*}}{\rightarrow} & [S_r S^{2n}, BH] & \overset{\psi_H}{\rightarrow} & k_H(S^{2n+1})
\end{array}$$

where k_K, f_*, Bf_*, and ψ_K have been defined before, ΩBK is the loop space of BK, and $f_\#$, $\mu_{K\#}$, $\Omega Bf_\#$, and adj_{K*} are given by $f_\#([\delta]) = [f \circ \delta]$, $\mu_{K\#}([\sigma]) = [\mu_K \circ \sigma]$($\mu_K$ is defined below), $\Omega Bf_\#([\kappa]) = [\Omega Bf \circ \kappa]$ with ΩBf: $\Omega BH \rightarrow \Omega BG$ given by $\Omega Bf(\gamma) = Bf \circ \gamma$, and $adj_{K*}([\alpha]) = [adj_k(\alpha)]$ with $adj_K(\alpha)([z, t]) = \alpha(z)(t)$, $t \in [0, 1]$. The set $[S^{2n}, K] = \Pi_{2n}(K)$ corresponds to the characteristic maps for the K-principal bundles over S^{2n+1}.

Proof. The commutativity of the right square has been proved in Section 4.2, with $S^{2n+1} = X$. The natural equivalence adj_K is given by the *exponential law* in function spaces (Spanier, 1966). By Switzer (1975) there exist *homotopy equivalences* μ_K such that the diagram

$$\begin{array}{ccc}
H & \overset{f}{\rightarrow} & G \\
\mu_H \uparrow & & \uparrow \mu_G \\
\Omega BH & \overset{\Omega Bf}{\rightarrow} & \Omega BG
\end{array}$$

commutes up to homotopy, therefore $\mu_{K\#}$ is a bijection and the first square commutes. Finally, notice that in the diagram of the proposition we are dealing with based homotopy classes of maps. This corresponds to based principal bundles. However, since we are taking path-connected topological groups, the function that forgets the base points is a bijection between based bundles and the usual unbased bundles of Section 4.2. QED

Theorem. For even n, $n \geq 2$, the clutching map g_{n+1} of the principal bundle $SU(n) \rightarrow SU(n+1) \overset{\pi_{n+1}}{\rightarrow} S^{2n+1}$ is a homotopy lifting of the $(2n-3)$th reduced suspension of the Hopf map h. For odd n, $n \geq 3$, $\pi_n \circ g_{n+1}$ is null-homotopic.

Proof. We apply the previous proposition to the case $H = SU(n)$, $G = U(n)$, and $f = \iota$ (the inclusion), for $n = 2, 3, \ldots$. By the proposition in Section 4.3, $[\xi] = \iota_*([\eta])$ with ξ: $U(n) \rightarrow U(n+1) \overset{p_{n+1}}{\rightarrow} S^{2n+1}$ and η: $SU(n) \rightarrow SU(n+1) \overset{\pi_{n+1}}{\rightarrow} S^{2n+1}$. Then one has the commutative diagram

$$\begin{array}{ccccccc}
[T'_{n+1}] \in & [S^{2n}, U(n)] & \overset{\mu}{\rightarrow} & k_{U(n)}(S^{2n+1}) & \ni [\xi] \\
& \iota_\# \uparrow & & \iota_* \uparrow & \\
[g_{n+1}] \in & [S^{2n}, SU(n)] & \overset{\nu}{\rightarrow} & k_{SU(n)}(S^{2n+1}) & \ni [\eta]
\end{array}$$

where $\mu = \psi_{U(n))} \circ adj_{U(n)*} \circ \mu_{U(n)\#}$ and $\nu = \psi_{SU(n)} \circ adj_{SU(n)*} \circ \mu_{SU(n)\#}$ are bijections, and T'_{n+1} is the clutching map for the principal bundle ξ (Steenrod, 1951). Then $[T'_{n+1}] = \mu^{-1}([\xi]) = \mu^{-1}(\iota_*([\eta])) = \mu^{-1} \circ \iota_*(\nu([g_{n+1}])) = \mu^{-1} \circ \iota_* \circ \nu([g_{n+1}]) = \iota_\#([g_{n+1}]) = [\iota \circ g_{n+1}]$ and therefore $T'_{n+1} \sim \iota \circ g_{n+1}$.

Consider the following diagram:

$$\begin{array}{ccc} & U(n+1) & \\ & \downarrow & \\ & U(n) & \\ T'_{n+1}\nearrow & & \searrow p_n \\ S^{2n} & \overset{S_r^{2n-3}h}{\rightarrow} & S^{2n-1} \end{array}$$

Steenrod (1951) proved that for n even, $n \geq 2$, $p_n \circ T'_{n+1} \sim S_r^{2n-3}h$, i.e., the diagram commutes up to homotopy, while for n odd, $n \geq 3$, $p_n \circ T'_{n+1} \sim$ const. Then, $p_n \circ T'_{n+1} \sim p_n \circ (\iota \circ g_{n+1}) = (p_n \circ \iota) \circ g_{n+1} = \pi_n \circ g_{n+1}$

$$\sim \begin{cases} S_r^{2n-3}h, & n \text{ even} \quad \text{QED} \\ \text{const}, & n \text{ odd} \end{cases}$$

Corollary. (i) If n is odd ($n \geq 3$), then $SU(n)$, as a principal $SU(n-1)$-bundle, does not have a reduction to the subgroup $SU(n-2) \overset{\iota}{\rightarrow} SU(n-1)$. (ii) If n is even ($n \geq 4$), then $SU(n)$, as a principal $SU(n-1)$-bundle, has a reduction to the subgroup $SU(n-2) \overset{\iota}{\rightarrow} SU(n-1)$.

Proof. By the proposition above we have the following commutative diagram:

$$\begin{array}{ccc} \Pi_{2n-2}(SU(n-2)) & \overset{\iota_\#}{\rightarrow} & \Pi_{2n-2}(SU(n-1)) \\ \updownarrow & & \updownarrow \\ k_{SU(n-2)}(S^{2n-1}) & \overset{\iota_*}{\rightarrow} & k_{SU(n-1)}(S^{2n-1}) \end{array}$$

$SU(n)$, as a principal $SU(n-1)$-bundle, is represented by the class $[\pi_n] \in k_{SU(n-1)}(S^{2n-1})$, and $[\pi_n]$ has a reduction to the subgroup $SU(n-2)$ if and only if $[\pi_n] \in \mathrm{Im}(\iota_*)$. Since the diagram commutes, this happens if and only if the homotopy class of its characteristic map $[g_n] \in \Pi_{2n-2}(SU(n-1))$ is in the image of $\iota_\#$. By the exact homotopy sequence of the bundle $SU(n-2) \rightarrow SU(n-1) \overset{\pi_{n-1}}{\rightarrow} S^{2n-3}$,

$$\ldots \rightarrow \Pi_{2n-2}(SU(n-2)) \overset{\iota_\#}{\rightarrow} \Pi_{2n-2}(SU(n-1)) \overset{\pi_{n-1\#}}{\rightarrow} \Pi_{2n-2}(S^{2n-3}) \rightarrow \ldots$$

we have that $\mathrm{Im}(\iota_\#) = ker(\pi_{n-1\#})$. Therefore $[\pi_n]$ has a reduction to $SU(n-2)$ if and only if $\pi_{n-1} \circ g_n$ is null-homotopic.

Let n be even; then by the theorem above, $\pi_{n-1} \circ g_n$ is null-homotopic and hence $[\pi_n]$ has a reduction. Let n be odd; by the theorem above, $\pi_{n-1} \circ g_n \simeq S_r^{2n-5}h$, which is not null-homotopic (Steenrod and Epstein, 1962); therefore $[\pi_n]$ has no reduction. QED

Proposition. $SU(n)$ as a principal $SU(n-1)$-bundle is not trivial, for all $n \geq 3$; in particular $SU(n)$ is not homeomorphic to $SU(n-1) \times S^{2n-1}$.

Proof. Let n be odd and assume that $SU(n)$ is a trivial bundle. Then this is equivalent to a reduction to the trivial subgroup $\{0\} \to SU(n-1)$, and this implies a reduction to any subgroup of $SU(n-1)$, in particular to $SU(n-2)$, which contradicts the corollary above.

Let n be even and assume that $SU(n)$ is a trivial bundle. Then $SU(n) \cong SU(n-1) \times S^{2n-1}$. By (Itô, 1993), $\Pi_{2n}(SU(n)) \cong \mathbb{Z}_{n!}$, $\Pi_{2n}(S^{2n-1}) \cong \mathbb{Z}_2$ and, since n is even, $\Pi_{2n}(SU(n-1)) \cong \mathbb{Z}_{n!/2}$. Therefore we would have that $\mathbb{Z}_{n!} \cong \mathbb{Z}_{n!/2} \oplus \mathbb{Z}_2$, but this implies that $n!/2$ and 2 are relatively prime, which is a contradiction. QED

5. S^2 AND RELATIVITY

As is well known, the Lorentz group, the group of linear transformations of Minkowski space-time which preserves the scalar product $\langle x, y\rangle = x^T\eta y$, where

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is the Minkowskian metric, is a subgroup of the symmetry group of several gauge theories of gravity (Hehl *et al.*, 1976; Basombrío, 1980). This means that $O(3, 1)$ is a subgroup of the structure group of the corresponding principal bundles. The relationship between these theories and the 2-sphere (the Riemann sphere $\mathbb{C}\cup\{\infty\}$) comes from the fact that there is a canonical Lie group isomorphism between the connected component of $O(3, 1)$, the proper orthochronous Lorentz group $SO^0(3, 1)$, and the group of conformal (Möbius) transformations of S^2, *Conf* (S^2). We recall that *Conf* (S^2) is the set of all invertible transformations of the Riemann sphere which preserves the angles between curves and that at each point multiply all the tangent vectors by a fixed positive number.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $GL_2(\mathbb{C})$; we define a Möbius transformation m: $S^2 \to S^2$ as follows: if $c \neq 0$, then

$$z \mapsto \begin{cases} \dfrac{az+b}{cz+d} & \text{if} \quad z \neq -d/c \\ \infty & \text{if} \quad z = -d/c \end{cases}$$

and

$$\infty \mapsto a/c$$

And if $c = 0$, then

$$\begin{cases} z \mapsto \dfrac{a}{d} z + \dfrac{b}{d} \\ \infty \mapsto \infty \end{cases}$$

It is then easy to verify that the following diagram commutes:

$$\begin{array}{ccc} & \mathbb{Z}_2 & \\ & \downarrow & \\ & SL_2(\mathbb{C}) & \\ \psi\swarrow & & \searrow\lambda \\ SO^0(3,1) & \xrightarrow{\tau} & Conf(S^2) \end{array}$$

where (i) the projections ψ and λ are two-to-one group homomorphisms, respectively given by

$$\psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$

$$\begin{pmatrix} \dfrac{|a|^2+|b|^2+|c|^2+|d|^2}{2} & \mathrm{Re}(a\bar{b}+c\bar{d}) & \mathrm{Im}(a\bar{b}+c\bar{d}) & \dfrac{|a|^2-|b|^2+|c|^2-|d|^2}{2} \\ \mathrm{Re}(a\bar{c}+b\bar{d}) & \mathrm{Re}(a\bar{d}+b\bar{c}) & \mathrm{Im}(a\bar{d}-b\bar{c}) & \mathrm{Re}(a\bar{c}-b\bar{d}) \\ -\mathrm{Im}(a\bar{c}+b\bar{d}) & \mathrm{Im}(a\bar{d}-b\bar{c}) & \mathrm{Re}(a\bar{d}-b\bar{c}) & -\mathrm{Im}(a\bar{c}-b\bar{d}) \\ \dfrac{|a|^2+|b|^2-|c|^2-|d|^2}{2} & \mathrm{Re}(a\bar{b}-c\bar{d}) & \mathrm{Im}(a\bar{b}-c\bar{d}) & \dfrac{|a|^2-|b|^2-|c|^2+|d|^2}{2} \end{pmatrix}$$

with

$$\psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \psi\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = l$$

(Penrose and Rindler, 1984), and $\lambda(g/\sqrt{detg}) = m$ with $\lambda(g/\sqrt{detg}) = \lambda(-g/\sqrt{detg})$; and (ii) $\tau(l) = m$ is the desired isomorphism. $SL_2(\mathbb{C}) \xrightarrow{\psi} SO^0(3,1)$ and $SL_2(\mathbb{C}) \xrightarrow{\lambda} Conf(S^2)$ are $\mathbb{Z}_2$-principal bundles.

Thus we conclude that the symmetry group of the standard model G'_{SM}, when gravitation is included, locally contains, as a space, $S^1 \times (S^3)^2 \times S^5 \times Conf(S^2)$.

Remark. In the framework of the theory of categories, functors, and natural transformations, some of the geometrical objects of the previous sections, e.g., spheres and the Hopf map, have a natural origin. This suggests a possible relation between symmetries in nature, and therefore conservation laws, and some of the most general mathematical concepts. The basic idea is that of a representable functor (Aguilar and Socolovsky, 1997b).

ACKNOWLEDGMENT

We thank a referee for helpful comments.

REFERENCES

Aguilar, M. A., and Socolovsky, M. (1997a). Reductions and extensions in bundles and homotopy, *Advances in Applied Clifford Algebras*, **7(S)**, 487–494.

Aguilar, M. A., and Socolovsky, M. (1997b). Naturalness of the space of states in quantum mechanics, *International Journal of Theoretical Physics*, **36**, 883–921.

Ashtekar, A., and Schilling, T. A. (1995). Geometry of quantum mechanics, *AIP Conference Proceedings* , **342**, 471–478.

Basombrío, F. G. (1980). A comparative review of certain gauge theories of the gravitational field, *General Relativity and Gravitation*, **12**, 109–136.

Choquet-Bruhat, Y., and DeWitt-Morette, C. (1989). *Analysis, Manifolds and Physics, Part II: 92 Applications*, North-Holland, Amsterdam.

Corichi, A., and Ryan, Jr., M. P. (1997). Quantization of nonstandard Hamiltonian systems, *Journal of Physics A: Mathematical and General*, **30**, 3553–3572.

Gell-Mann, M., and Ne'eman, Y. (1964). *The Eightfold Way*, Benjamin, New York.

Greub, W., Halperin, S., and Vanstone, R. (1973). *Connections, Curvature, and Cohomology*, Academic Press, New York.

Hehl, F. W., von der Heyde, P., and Kerlick, G. D. (1976). General relativity with spin and torsion: Foundations and prospects, *Reviews of Modern Physics*, **48**, 393–416.

Itô, K., ed. (1993). *Encyclopedic Dictionary of Mathematics*, Mit Press, Cambridge, Massachusetts.

Kobayashi, S., and Nomizu, K. (1963). *Foundations of Differential Geometry*, Vol. I, Wiley, New York.

Mohapatra, R. N. (1986). *Unification and Supersymmetry*, Springer-Verlag, New York.

Naber, G. L. (1998). Private communication.

Penrose, R., and Rindler, W. (1984). *Spinors and Space-Time*, Vol. 1, Cambridge University Press, Cambridge.

Saller, H. (1998). External-internal group quotient structure for the standard model in analogy to general relativity, *International Journal of Theoretical Physics*, **37**, 2333–2361.

Spanier, E. H. (1966). *Algebraic Topology*, Springer-Verlag, New York.

Steenrod, N. (1951). *The Topology of Fibre Bundles*, Princeton University Press, Princeton, New Jersey.

Steenrod, N., and Epstein, D. B. A. (1962). *Cohomology Operations*, Princeton University Press, Princeton, New Jersey.

Switzer, R. (1975). *Algebraic Topology—Homotopy and Homology*, Springer-Verlag, New York.

Taylor, J. C. (1976). *Gauge Theories of Weak Interactions*, Cambridge University Press, Cambridge.

Trautman, A. (1984). *Differential Geometry for Physicists*, Bibliopolis, Naples.

Wu, T. T., and Yang, C. N. (1975). Concept of nonintegrable phase factors and global formulation of gauge fields, *Physical Review D*, **12**, 3845–3857.